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UNILATERAL CONTACT, FRICTION AND RELATED INTERACTIONS IN CRACKS. THE DIRECT BOUNDARY INTEGRAL METHOD

P. S. THEOCARIS
National Academy of Athens, P.O. Box 77230, 17510 Athens, Greece

and

P. D. PANAGIOTOPOULOS† and C. BISBOS
Department of Civil Engineering, Aristotle University, 54006 Thessaloniki, Greece

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Abstract—The paper presents a theory for the study of cracks having a given geometry by taking into account all types of actions of monotone type like unilateral contact and friction phenomena between the two crack sides. The arising problems are of a non-classical nature, due to the interface conditions expressed in terms of non-differentiable convex superpotentials. The direct B.I.E.M. is extended appropriately in order to treat this type of problem. The developed method is illustrated by a numerical example concerning the calculation of stress intensity factors under the unilateral contact and friction interface conditions.

1. INTRODUCTION

Unilateral contact and friction give rise to highly non-linear problems due to the *a priori* unknown free boundaries between contact and non-contact regions. These two phenomena have been extensively studied for deformable bodies both from the mathematical and the numerical point of view [cf. Duvaut and Lions (1972), Panagiotopoulos (1985), and the references given therein]. The inequalities describing unilateral contact and friction cause the unilateral character of the corresponding mechanical problems, since for these problems the "principles" of virtual and complementary virtual works hold only in inequality form. Moreover, the problems are no longer expressed in terms of differential equations, but of multivalued differential equations, which are equivalent to variational inequalities expressing the "principle" of virtual, or of complementary virtual, work. The same situation arises in the more general case where we have boundary or interface conditions expressed in terms of the convex superpotentials of Moreau (1968) [cf. also Panagiotopoulos (1985)], i.e. conditions of monotonic multivalued type. The existing cracks in solids form interfaces which present unilateral contact effects in the normal direction and frictional effects in the tangential direction to the interface. This non-classical behaviour of cracks has not as yet been extensively studied. We mention the works by Comninou (1977) and by Dundurs and Comninou (1979) which concern especially the frictional contact in cracks where an unnecessary assumption concerning the contact zone is made [cf. in this respect also Dubourg (1989), Panagiotopoulos (1975), and Panagiotopoulos and Talaslidis (1980)]. Cracks with the unilateral contact effect have also been recently considered in Zang and Gudmundson (1990), where an incremental trial and error method is developed based on the boundary integral equation (B.I.E.) method for the determination of the contact and non-contact regions within the crack. After a few iterations, during which the contact configuration within the crack is changing, the correct solution is found, if the load increments are small enough. Unilateral contact problems for cracks including friction have been studied by the indirect boundary integral equation method (B.I.E.M.) by Dubourg (1989), Dubourg *et al.* (1988), and Theocaris and Panagiotopoulos (1992). Finally, we also mention the work of Bower (1987) which uses techniques similar to the techniques of Dubourg (1989).

All these solutions, except those of Theocaris and Panagiotopoulos (1992), concentrate on the crack related questions, as e.g. the arising singularities and their numerical treatment,

† Also at: Faculty of Mathematics and Physics, RWTH, 5100 Aachen, Germany.

the calculation of the stress intensity factors, etc. In Theocaris and Panagiotopoulos (1992) we have tried to optimally exploit the inequality nature of the problem according to the recent numerical approaches to inequality problems. Therefore we have combined the indirect B.I.E.M. for the crack modelling with the algorithm of Bisbos (1991), which is convenient for large scale 3D-unilateral contact problems and has already been successfully tested for the solution of industrial applications in the automobile industry. However, the use of the indirect B.I.E.M. has certain disadvantages concerning the numerical treatments of the singularities. Moreover it does not give rise to minimum problems, a fact which does not permit the method to reach the full automation level of the numerical methods for inequality problems, which are particularly benefitted by the corresponding developments in optimization algorithms.

The aim of the present paper is to develop the direct B.I.E.M. for the same problem, i.e. for a crack with unilateral contact and friction interface conditions, and more generally for the case of interactions of the two crack sides of monotone multivalued nature. The presented method here concerns cracks with given length, which do not change during the loading procedure. The direct B.I.E.M. method applied here is an extension to the crack problems of the direct B.I.E.M. for problems having unilateral boundary conditions (Panagiotopoulos and Lazaridis, 1987; Panagiotopoulos, 1987; Antes and Pangiotopoulos, 1992). In these works we have used the duality and the Lagrangian theories of convex analysis. The problems lead to multivalued boundary integral equations which are equivalent to minimum problems formulated along the boundaries with respect to the inequality constrained reactions or displacements, i.e. all bilateral degrees of freedom are eliminated [cf. in this context also Kalker and van Randen (1972) and Bufler (1985)].

Here we have developed another method using Betti's theorem in order to avoid the use of duality and of Lagrangian analyses, which in the presence of singularities present serious problems. Indeed there are two reasons why the duality of optimization problems and the related theory of Lagrangians and saddle points should be avoided in the case of domains with cuts, as in crack problems. The duality theory, as it is formulated in Ekeland and Temam (1976) and Panagiotopoulos (1985) assumes that the domain of the problem Ω is "appropriately regular" and, in the case of an elastic body, that $\sigma_{ij} \in L^2(\Omega)$ and $\sigma_{ij,j} \in L^2(\Omega)$. In the case of cracks we have a non-regular domain and a stress singularity at the crack tip. In a domain with a cut several of the properties of Sobolev spaces, which is the classical function space of elastic bodies with regular boundaries, do not hold, as e.g. the density property of $C^{\infty}(\bar{\Omega})$-functions to the Sobolev space $H^1(\Omega)$ (Grisvard, 1985), or need special care, as is the case with the "trace" properties, which are crucial to the crack contact problem treated here. In this respect many unanswered mathematical questions have existed until now, even for the simple case of circular domains with one cut, studied in Grisvard (1985). Note that the general duality theory of Ekeland and Temam (1976) holds for very general functional spaces and thus one should be able to formulate a duality theory of variational principles allowing for the stress singularities, under the condition that the equations of elasticity are well-posed in such a framework. In other words one should guarantee, using advanced functional analysis, that all the integrals arising in the application of the duality theory have a meaning for each type of domain and crack. This is a difficult task which still leaves several unanswered mathematical questions. The situation becomes more delicate in the case of a slightly more complicated crack and/or domain geometry. For this reason we have avoided the method of duality and we have applied Betti's theorem which has a broader validity holding both for the forces being functions or Dirac measures (unit forces) and for any type of domain and crack geometry. The numerical implementation of the direct B.I.E.M. necessitates the use of some special elements for the consideration of the arising crack singularities.

2. CRACKS WITH UNILATERAL CONTACT AND FRICTION. OTHER TYPES OF INTERFACE CONDITIONS

Let us consider a three-dimensional linear elastic body Ω, which is assumed to occupy an open bounded subset of $\mathbb{R}^3$ in its undeformed state and has a regular boundary Γ. We

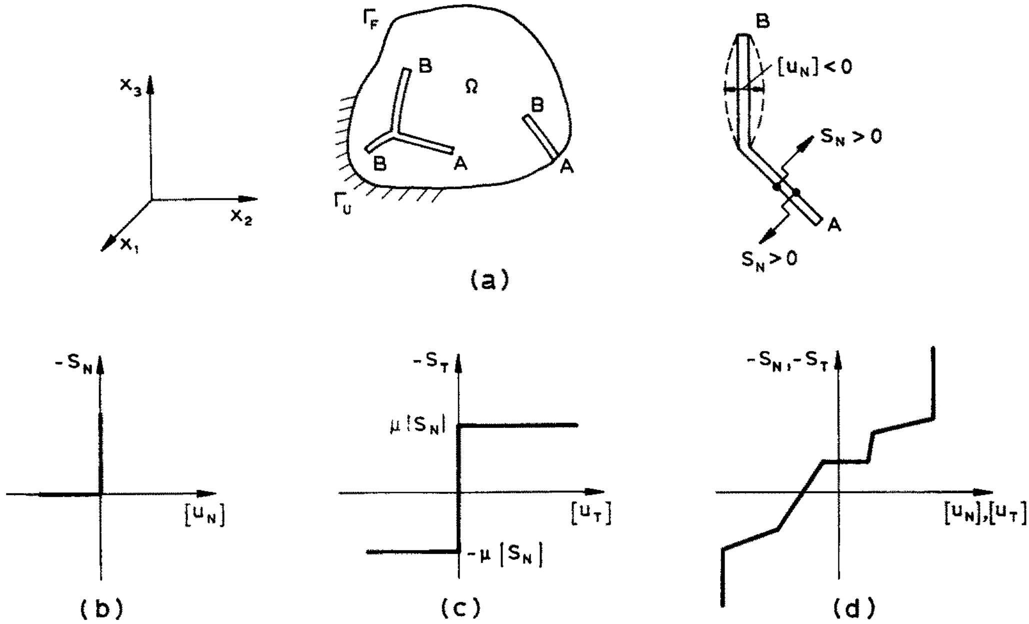


Fig. 1. Geometry of the body and the crack interfaces. Possible interface laws.

refer Ω to a Cartesian orthogonal coordinate system $Ox_1x_2x_3$ [Fig. 1(a)]. Γ is decomposed into two mutually disjoint parts Γ_U and Γ_F. It is assumed that on Γ_U (resp. Γ_F) the displacements (resp. the tractions) are given. Let $n = \{n_i\}$ be the outward unit normal vector to Γ and $S = \{S_i\} = \{\sigma_{ij}n_j\}$ the traction vector on the boundary, where $\sigma = \{\sigma_{ij}\}$ is the stress tensor. We denote by $u = \{u_i\}$ the displacement vector, by $\varepsilon = \{\varepsilon_{ij}\}$ the strain tensor (small strain assumption) and by $C = \{C_{ijhk}\}, i, j, h, k = 1, 2, 3$ Hooke's tensor of elasticity obeying the well-known symmetry and ellipticity conditions.

On Γ_U we have, for the sake of simplicity, the homogeneous conditions

$$u_i = 0, \tag{1}$$

otherwise we have to make the problem homogeneous through a translation. On Γ_F the conditions

$$S_i = F_i, \quad F_i = F_i(x), \tag{2}$$

hold. Within the body Ω some formed cracks are given which are denoted by AB. They may have any geometrical shape and their contour does not change during the loading. We assume unilateral contact and friction conditions holding at the interface. In order to define them we consider the components S_N and $S_T = \{S_{T_i}\}$ of the boundary tractions $S = \{S_i\}$ normally and tangentially to Γ. They read

(i) if $[u_N] < 0$ then $S_N = 0$ and $S_T = \{S_{T_i}\} = 0$, (3)

(ii) if $[u_N] = 0$ then $S_N \leqslant$ and (4)

(a) if $[S_T] < \mu|S_N|$ then $[u_T] = \{[u_{T_i}]\} = 0$ and (5)

(b) if $|S_T| = \mu|S_N|$, then exists $\lambda \geqslant 0$ such that $[u_{T_i}] = -\lambda S_{T_i}, \quad i = 1, 2, 3.$ (6)

Here $[u_N]$ and $[u_T] = \{[u_{T_i}]\}$ are the relative displacements of the two crack sides. We consider $[u_N]$ as negative if the crack tends to open. Moreover μ is the friction coefficient.

Note that (3) [resp. (4)] is fulfilled at those points of the crack interface which do not remain (resp. are) in contact, whereas (5) [resp. (6)] corresponds to the region of adhesive (resp. of sliding) friction. Concerning now the behaviour of the body we assume that it is a linear elastic body obeying Hooke's law and that we have a geometrically linear theory. The problem we want to solve reads:

Problem. Find for a given loading f the stress, strain and displacement fields. Moreover the contact and non-contact regions and the adhesive and sliding friction regions should be determined within each crack. As it is well known (Panagiotopoulos, 1985; Moreau, 1968; Ekeland and Temam, 1976), (3)–(6) can be put in the following subdifferential form:

$$-S_N \in \partial I_K([u_N]); \tag{7}$$

$$\text{if } [u_N] < 0, \quad \text{then} \quad S_T = 0 \quad \text{otherwise} \tag{8}$$

$$-\{S_{T_i}\} \in \partial(\mu|S_N|\,|[u_T]|). \tag{9}$$

Here $K = \{S_N | S_N \leqslant 0\}$, $I_K = \{0 \text{ if } S_N \in K, \infty \text{ otherwise}\}$ and ∂ denotes the subdifferential of convex analysis. We recall here that for a convex function $\phi : \mathbb{R}^n \to (-\infty, +\infty], \phi \not\equiv \infty$ holds by definition that

$$\partial\phi(x) = \{x_1^* | \phi(x^*) - \phi(x) \geqslant (x_1, x^* - x)\ \forall x \in \mathbb{R}^n\}, \tag{10}$$

where (,) denotes the $\mathbb{R}^n$-inner product, and the set of points $\{x_1, \partial\phi(x)\}$ defines a monotone possibly multivalued graph. For instance, in the $\mathbb{R}^1$-case this graph is monotone and includes filled-in vertical finite jumps or infinite jumps to the left and/or to the right. Then ϕ is called a superpotential at the (x_1^*, x) law. The conditions (7)–(9) can be put in the general implicit form:

$$-S_N \in \partial j_N([u_N]; S_T), \quad -S_T \in \partial j_T([u_T]; S_N), \tag{11}$$

where j_N and j_T are the corresponding superpotentials and ∂ is taken with respect to $[u_N]$ and $[u_T]$, respectively. With the scope to apply the decoupling iterative method introduced in Panagiotopoulos (1975) we assume for the present that the normal interaction of the two crack sides is independent from the tangential interaction. Therefore we shall formulate the B.I.E.M. assuming that on the crack interfaces the laws [Fig. 1(d)]

$$-S_N \in \partial j_N([u_N]), \tag{12}$$

$$-S_T \in \partial j_T([u_T]), \tag{13}$$

hold, or equivalently their inverse laws

$$[u_N] \in \partial j_N^c(-S_N), \tag{12a}$$

$$[u_T] \in \partial j_T^c(-S_T), \tag{13a}$$

where j_N^c (resp. j_T^c) is the conjugate function (Panagiotopoulos, 1985; Moreau, 1968), of j_N (resp. j_T). The equations of the boundary value problem (B.V.P.) read

$$\sigma_{ij,j} + f_i = 0 \quad \text{in } \Omega, \tag{14}$$

$$\varepsilon_{ij} = \varepsilon_{ij}(u) = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{in } \Omega, \tag{15}$$

$$\sigma_{ij} = C_{ijhk}\varepsilon_{hk} \quad \text{in } \Omega, \tag{16}$$

where the comma denotes the partial derivation and $f = \{f_i\}$ is the volume force vector. Now let V be the linear space of the displacements v_i and let V_0 be the set of the kinematically admissible displacements, i.e.

$$V_0 = \{v | v = \{v_i\}, \quad v_i \in V_i, \quad v_i = 0, \quad i = 1,2,3 \text{ on } \Gamma_U\}. \tag{17}$$

The work $\int f_i v_i \, d\Omega$ of the force $f = \{f_i\}$ for the displacement $v = \{v_i\}$ is denoted by (f, v), and the bilinear form of elasticity by

$$a(u,v) = (C\varepsilon(u), \varepsilon(v)) = \int_\Omega C_{ijhk}\varepsilon_{hk}(u)\varepsilon_{hk}(v)\, d\Omega. \tag{18}$$

Let us finally introduce the notation

$$l(v) = (f, v) + \int_{\Gamma_F} F_i v_i \, d\Gamma. \tag{19}$$

3. FORMULATION WITH RESPECT TO THE CRACK INTERFACE TRACTIONS

Let L be the admissible space for the tractions S on each crack AB. We denote further all the crack interfaces $\cup_{AB}$ by γ.

First we assume that $S = \{S_i\}$ is given on γ and is equal to $\mu = \{\mu_i\}$. Then the solution of the arising classical problem satisfies the following problem: Find $u = u(\mu) \in V_0$ such that

$$a(u,v) - \int_\gamma \mu_i [v_i]\, d\Gamma - (f, v) - \int_{\Gamma_F} F_i v_i \, d\Gamma = 0 \quad \forall v \in V_0. \tag{20}$$

Obviously (20) expresses the principle of virtual work for a structure resulting from the initial one by eliminating the superpotential constraints on γ and by applying the corresponding forces $\mu = \{u_i\}$. The bilinearity of (20) implies that u is a linear function of μ_i, f and F_i. Thus the solution u of (20) can be written as the sum of $\tilde{u}_{(1)} \in V_0$ and $\tilde{u}_{(2)} \in V_0$, where $\tilde{u}_{(1)}$ and $\tilde{u}_{(2)}$ are solutions of the two variational equalities

$$a(\tilde{u}_{(1)}, v) - l(v) = 0 \quad \forall v \in V_0 \tag{21}$$

and

$$a(\tilde{u}_{(2)}, v) - \int_\gamma \mu_i [v_i]\, d\Gamma = 0 \quad \forall v \in V_0, \tag{22}$$

respectively. Here $\tilde{u}_{(1)}$ and $\tilde{u}_{(2)}$ are equilibrium configurations of two classical (bilateral) structures which are obtained from the initial one by ignoring the superpotential conditions on γ and assuming that on certain parts of the boundary the load is zero.

Thus in (21) the structure is loaded by the forces f on Ω and F on Γ_F, whereas on γ the loading is zero. Moreover the structure is fixed on Γ_U. In (22) the structure is loaded by a force $\mu = \{u_i\}$ on γ only and is fixed along Γ_U; the loading in Ω and Γ_F is zero. The solutions $\tilde{u}_{(1)}$ and $\tilde{u}_{(2)}$ exist and are unique, as it is well known from the classical theory of elasticity. For these bilateral structures the solution $\tilde{u}_{(1)}$ and $\tilde{u}_{(2)}$ can be written in terms of Green's operator G, which is the same for both structures because in each case the same type of boundary conditions hold. Thus we have

$$\tilde{u}_{(1)} = G(\tilde{l}), \quad \tilde{u}_{(2)} = G(\mu), \quad u = \tilde{u}_{(1)} + \tilde{u}_{(2)}, \quad \tilde{l} = \{f, F\}. \tag{23}$$

It remains to determine the unknown distribution $\mu = \{\mu\}$ on γ. With respect to the linear elasticity problem corresponding to (22) Betti's theorem is applied: Assume that $\lambda = \{\lambda_i\}$ on γ is a force distribution corresponding to a displacement field $v_{(2)} \in V_0$ if $f = 0$ and $F = 0$ on Γ_F. Then we have

$$\int_\gamma \lambda_i[\tilde{u}_{(2)i}]\,d\Gamma = \int_\gamma \mu_i[v_{(2)i}]\,d\Gamma, \tag{24}$$

and we may write that

$$v_{(2)} = G(\lambda). \tag{25}$$

Now (24) together with (25) implies that for every $\lambda \in L$

$$\begin{aligned} \int_\gamma \lambda_i[\tilde{u}_i]\,d\Gamma &= \int_\gamma \lambda_i[\tilde{u}_{(1)i}]\,d\Gamma + \int_\gamma \lambda_i[\tilde{u}_{(2)i}]\,d\Gamma \\ &= \int_\gamma \lambda_i[u_{(1)i}]\,d\Gamma + \int_\gamma \mu_i[v_{(2)i}]\,d\Gamma \\ &= \int_\gamma \lambda_i[(G(\tilde{l}))]_i\,d\Gamma + \int_\gamma \mu_i[(G(\lambda))]_i\,d\Gamma. \end{aligned} \tag{26}$$

Let us introduce now the bilinear form

$$\beta(\lambda, \mu) = \int_\gamma \mu_i[(G(\lambda))]_i\,d\Gamma \tag{27}$$

which is symmetric by Betti's theorem and the linear form

$$b(\lambda) = -\int_\gamma \lambda_i[(G(\tilde{l}))]_i\,d\Gamma. \tag{28}$$

Assuming now that the tractions $\mu = (\mu_N, \mu_T)$ on Γ_S are related to the displacement field u through the relations (12), (13) we may write using the definition (10) of the subdifferential that

$$\begin{aligned} J_N^c(-\lambda_N^*) - J_N^c(-\mu_N) + J_T^c(-\lambda_N^*) - J_T^c(-\mu_T) \geqslant -[u_N](\lambda_N^* - \mu_N) \\ -[u_T]_i(\lambda_{T_i}^* - \mu_{T_i})\ \forall \lambda^* \in L. \end{aligned} \tag{29}$$

From (29) and (26) we get for $\lambda^* \equiv \lambda$ that for $\mu \in L$:

$$J_N^c(-\lambda_N) - J_N^c(-\mu_N) + J_T^c(-\lambda_T) - J_T^c(-\mu_T) \geqslant b(\lambda - \mu) - \beta(\mu, \lambda - \mu)\ \forall \lambda \in L, \tag{30}$$

where

$$J_{\mathrm{N}}^{\mathrm{c}}(-\lambda_{\mathrm{N}}) = \begin{cases} \int_{\gamma} j_{\mathrm{N}}^{\mathrm{c}}(-\lambda_{\mathrm{N}})\,\mathrm{d}\Gamma & \text{if the integral exists,} \\ \infty & \text{otherwise.} \end{cases} \tag{31}$$

Analogous is the definition of $J_{\mathrm{T}}^{\mathrm{c}}$. Thus we are led to the following variational inequality: Find $\mu = \{\mu_{\mathrm{N}}, \mu_{\mathrm{T}}\} \in L$ in order to satisfy

$$\beta(\mu, \lambda-\mu) - b(\lambda-\mu) + J_{\mathrm{N}}^{\mathrm{c}}(-\lambda_{\mathrm{N}}) - J_{\mathrm{N}}^{\mathrm{c}}(-\mu_{\mathrm{N}}) + J_{\mathrm{N}}^{\mathrm{c}}(-\lambda_{\mathrm{T}}) \\ - J_{\mathrm{T}}^{\mathrm{c}}(-\mu_{\mathrm{T}}) \geqslant 0 \quad \forall \lambda = (\lambda_{\mathrm{N}}, \lambda_{\mathrm{T}}) \in L. \tag{32}$$

4. FORMULATION WITH RESPECT TO THE CRACK INTERFACE RELATIVE DISPLACEMENTS

Let us assume first that $u_i = U_i$ on Γ_{U}. Finally we are free to take $U_i = 0$. We suppose that the relative displacements $[u]$ on γ are prescribed, let Σ be the set of all symmetric stress-tensors and let

$$\Sigma_1 = \{\tau | \tau = \{\tau_{ij}\}, \quad \tau_{ij} = \tau_{ji}, \quad \tau_{ij,j} + f_i = 0 \quad \text{in} \quad \Omega, \quad T_i = F_i \quad \text{on} \quad \Gamma_{\mathrm{F}}\} \tag{33}$$

be the statically admissible set. Here $\{T_i\}$ denotes the traction on Γ corresponding to the stress $\tau = \{\tau_{ij}\}$. Let also $\{c_{ijhk}\}$ be the inverse tensor to $C = \{C_{ijhk}\}$ i.e.

$$\varepsilon_{ij} = c_{ijhk}\sigma_{hk} \tag{34}$$

and let

$$A(\sigma, \tau) = (c\sigma, \tau) = \int_{\Omega} c_{ijhk}\sigma_{ij}\tau_{hk}\,\mathrm{d}\Omega. \tag{35}$$

If the relative displacements $[v]$ on γ are considered as given, the "principle" of complementary virtual work for the body takes the following form: Find $\sigma = \sigma([v]) \in \Sigma_1$ such that

$$A(\sigma, \tau) = \int_{\Gamma_{\mathrm{U}}} U_i T_i\,\mathrm{d}\Gamma + \int_{\gamma} [v_i] T_i\,\mathrm{d}\Gamma \quad \forall \tau \in \Sigma_1. \tag{36}$$

Let us now introduce a stress field $\sigma_0 \in \Sigma_1$, i.e. a stress field satisfying the equations of equilibrium and the static boundary conditions on Γ_{F}. We consider the new variables

$$\bar{\sigma} = \sigma - \sigma_0 \quad \text{and} \quad \bar{\tau} = \tau - \tau_0, \tag{37}$$

where σ, $\tau \in \Sigma_0$ and

$$\Sigma_0 = \{\tau | \tau = \{\tau_{ij}\}, \quad \tau_{ij} = \tau_{ji}, \quad \tau_{ij,j} = 0 \quad \text{in} \quad \Omega, \quad T_i = 0 \quad \text{on} \quad \Gamma_{\mathrm{F}}\}. \tag{38}$$

Thus (36) takes the form: Find $\sigma = \sigma(v) \in \Sigma_0$ such as to satisfy

$$A(\bar{\sigma}, \bar{\tau}) - \int_{\Gamma_{\mathrm{U}}} U_i \bar{T}_i\,\mathrm{d}\Gamma + \int_{\Gamma_{\mathrm{S}}} [v_i] \bar{T}_i\,\mathrm{d}\Gamma - A(\sigma_0, \tau) = 0 \quad \forall \tau \in \Sigma_0. \tag{39}$$

The Green–Gauss theorem implies that

$$
\begin{aligned}
A(\sigma_0, \tau) &= \int_\Omega \varepsilon_{0ij} \bar{T}_{ij}\, d\Omega \\
&= -\int_\Omega \tilde{u}_{0i} \bar{\tau}_{ij,j}\, d\Omega + \int_\gamma [\tilde{u}_{0i}] \bar{T}_i\, d\Gamma + \int_{\Gamma_F} \tilde{u}_{0i} \bar{T}_i\, d\Gamma + \int_{\Gamma_U} \tilde{u}_{0i} \bar{T}_i\, d\Gamma \\
&= \int_\gamma [\tilde{u}_{0i}] \bar{T}_i\, d\Gamma + \int_{\Gamma_U} \tilde{u}_{0i} \bar{T}_i\, d\Gamma \quad \forall \bar{\tau} \in \Sigma_0,
\end{aligned} \tag{40}
$$

where $\varepsilon_0 = c\sigma_0$ and $\tilde{u}_0$ is the displacement field corresponding to σ_0. For the determination of σ_0 the only condition fulfilled is the condition $\sigma_0 \in \Sigma_1$; we are free to choose any other type of kinematic or static boundary conditions on Γ_U and on γ. Thus, we could choose as the unique solution of a bilateral problem for an elastic body having on Γ_U and on γ zero displacements and subjected to forces f_i in Ω and F_i on Γ_F. Then (39) takes the form

$$
A(\sigma_0, \bar{\tau}) = 0 \quad \forall \bar{\tau} \in \Sigma_0. \tag{41}
$$

The stress $\bar{\sigma}$ in (39) is now written as the sum $\bar{\sigma}_{(1)} + \bar{\sigma}_{(2)}$ where $\bar{\sigma}_{(1)}$ and $\bar{\sigma}_{(2)}$ are solutions of the variational equalities respectively. Both (42) and (43) express the "principle" of complementary virtual work for bilateral elastic bodies

$$
A(\bar{\sigma}_{(1)}, \bar{\tau}) - \int_\gamma [v_i] \bar{T}_i\, d\Gamma = 0 \quad \forall \bar{\tau} \in \Sigma_0, \tag{42}
$$

$$
A(\bar{\sigma}_{(2)}, \bar{\tau}) - \int_{\Gamma_U} U_i \bar{T}_i\, d\Gamma = 0 \quad \forall \bar{\tau} \in \Sigma_0, \tag{43}
$$

resulting from the initial one in the following way: For the first (resp. the second) we consider the body Ω under the action of "prescribed" relative displacements $[v]$ (resp. zero) on γ, zero forces in Ω and Γ_F and zero (resp. U) displacements on Γ_U. Since these bodies are linear elastic, $\bar{\sigma}_{(1)}$ and $\bar{\sigma}_{(2)}$ are uniquely determined. Therefore from (42) and (43) we obtain that

$$
\bar{\sigma}_{(1)} = H([v]), \quad \bar{\sigma}_{(2)} = H(U), \quad \bar{\sigma} = \bar{\sigma}_{(1)} + \bar{\sigma}_{(2)}, \quad \bar{\sigma} \in \Sigma_0, \tag{44}
$$

where H is the stress-displacement operator of elasticity theory. The bilateral structures corresponding to (42) and (43) have the same H-operator since they are subjected to the same type of boundary conditions. Moreover let $\tilde{H}$ be the operator $N \circ H$ where N transforms σ into the boundary traction $S = \{S_i\} = \{\sigma_{ij} n_j\}$. Thus we may write that

$$
\bar{S}_{(1)} = \tilde{H}([v]), \quad \bar{S}_{(2)} = \tilde{H}(U). \tag{45}
$$

We have to determine the unknown relative displacement distribution $[v] = \{[v_i]\} \in \Lambda$ on γ; here Λ denotes the space of $[v]$. Let $[w] = \{[w_i]\} \in \Lambda$ be another relative displacement distribution on γ corresponding to the stress field $\bar{\tau}_{(1)} \in \Sigma_0$ through (42). Similarly $[v] = \{[v_i]\}$ corresponds to $\bar{\sigma}_{(1)} \in \Sigma_0$. Applying Betti's theorem we can write that

$$
\int_\gamma \bar{T}_{(1)i} [v]_i\, d\Gamma = \int_\gamma \bar{S}_{(1)i} [w]_i\, d\Gamma. \tag{46}
$$

Analogously to (45) we may write the relation

$$\bar{T}_{(1)} = \bar{H}([w]). \tag{47}$$

Relation (46) implies with (47) and (44) that

$$\begin{aligned}\int_\gamma [w]_i \bar{S}_i \,d\Gamma &= \int_\gamma [w]_i \bar{S}_{(1)i} \,d\Gamma + \int_\gamma [w]_i \bar{S}_{(2)i} \,d\Gamma \\ &= \int_\gamma [v]_i \bar{T}_{(1)i} \,d\Gamma + \int_\gamma [w]_i \bar{S}_{(2)i} \,d\Gamma \\ &= \int_\gamma [v]_i [\tilde{H}([w])]_i \,d\Gamma + \int_\gamma [w]_i [\tilde{H}(U)]_i \,d\Gamma. \end{aligned} \tag{48}$$

Now the bilinear form

$$\delta([v],[w]) = \int [\bar{H}([v])]_i [w]_i \,d\Gamma \tag{49}$$

and the linear form

$$\bar{d}([w]) = -\int_\gamma [\tilde{H}(U)]_i [w]_i \,d\Gamma \tag{50}$$

are introduced. Note that $\delta(.,.)$ is symmetric due to Betti's theorem. Thus (48) implies that

$$\int_\gamma [w]_i \bar{S}_i \,d\Gamma = \delta([v],[w]) - \bar{d}([w]). \tag{51}$$

But (12) and (13) imply, using (10), that

$$\begin{aligned} j_N([w_N^*]) - j_N([v_N]) - j_T([w_T^*]) - j_T([v_T]) &\geqslant -[S_N([w_N^*]_N - [v_N]) \\ &\qquad + S_{T_i}([w_{T_i}^*]_N - [v_{T_i}])] \\ &= -[S_i([w_i^*] - [v_i])] \\ &= -(S_i + S_{0i})([w_i^*] - [v_i]) \quad \forall [w^*] \in \Lambda. \end{aligned} \tag{52}$$

Setting $w^* \equiv w$, using the definition

$$J_N([v_N]) = \begin{cases} \int_\gamma j_N([v_N]) \,d\Gamma & \text{if the integral exists,} \\ \infty & \text{otherwise,} \end{cases} \tag{53}$$

and the analogous definition for $J_T([v_T])$, we obtain from (51) and (52) the following variational formulation: Find $[v] = \{[v_N],[v_T]\} \in \Lambda$ such as to satisfy the variational inequality

$$\begin{aligned}\delta([v],[w]-[v]) - d([w]-[v]) + J_N([w_N]) - J_N([v_N]) + J_T([w_T]) \\ - J_T([v_T]) \geqslant 0 \quad \forall [w] = \{[w_N],[w_T]\} \in \Lambda. \end{aligned} \tag{54}$$

Here

$$d([w]) = \bar{d}([w]) - \int_{\Gamma} S_{0i}[w_i]\,\mathrm{d}\Gamma. \tag{55}$$

5. MINIMUM PROBLEMS AND THE NUMERICAL TREATMENT

The derived variational inequalities (32) and (54) holding on the crack interface, are equivalent to minimum problems which are formulated in the sequel. Indeed using the fact that the bilinear forms $\beta(.,.)$ and $\delta(.,.)$ are generally non-negative and symmetric, we can show [cf. Kalker (1988)] the following results:

Proposition 5.1. Every solution of the variational inequality (32) [resp. (54)] solves the minimum problem

$$\Pi(\lambda) = \min\{\Pi(\mu)\,|\,\mu \in L\}, \tag{56}$$

$$(\text{resp.}\quad \tilde{\Pi}([v]) = \min\{\tilde{\Pi}([w])\,|\,[w] \in \Lambda\}), \tag{57}$$

where

$$\Pi(\lambda) = \tfrac{1}{2}\beta(\lambda,\lambda) - b(\lambda) + J_N^c(-\lambda_N) + J_T^c(-\lambda_T) \tag{58}$$

and

$$\tilde{\Pi}([v]) = \tfrac{1}{2}\delta([v],[v]) - d([v]) + J_N([v_N]) + J_T([v_T]). \tag{59}$$

Note here that using (10) we obtain that (32) is equivalent to the multivalued integral equation on γ

$$b - \tfrac{1}{2}\,\mathrm{grad}\,\beta(\lambda,\lambda) \in \partial(J_N^c(-\lambda_N) + J_T^c(-\lambda_T)). \tag{60}$$

Analogously (54) yields the multivalued integral equation (integral inclusion)

$$d - \tfrac{1}{2}\,\mathrm{grad}\,\delta([v],[v]) \in \partial(J_N([v_N]) + J_T([v_T])). \tag{61}$$

For the numerical calculation we must solve the arising multivalued B.I.E.s on γ, or equivalently the corresponding minimum problems by means of an appropriate optimization algorithm. The expression of $\delta(.,.)$ [resp. $\beta(.,.)$] is easily obtained for a discretized body by applying unit relative displacements (resp. unit forces) to each pair of corresponding nodes of the two crack faces and keeping all other pairs fixed (resp. leaving all other pairs free). Then the corresponding reactions of the fixed nodes obtained either by the classical F.E.M. or B.E.M. give the columns of the matrix corresponding to the bilinear $\delta(.,.)$. Analogously the relative displacements of the pairs of the free nodes give the columns of the matrix corresponding to $\beta(.,.)$; similarly the vectors d and b are obtained [cf. e.g. Panagiotopoulos and Lazaridis (1987)] by calculating the reactions of the fixed node pairs (resp. the relative displacements of the free node pairs) for the prescribed displacements (resp. for the given loading). Here, in order to take into account the crack singularity we must use special crack finite elements or boundary elements depending on the method used for the numerical calculation of the discretized problem. It is worth noting that generally the symmetry is lost in the discretized problem, with the exception of some special cases. This is made obvious e.g. for a curvilinear boundary, if one uses the classical B.I.E.M. for the calculation of the discretized bilinear and linear forms of the problem. In the case of lack of symmetry the variational inequalities are not equivalent to minimum problems. However, since the symmetry is predicted from Betti's theorem, we can consider the symmetrized problem, solve the corresponding minimum problem and then make some corrections by estimating the non-symmetric part and appropriately changing the linear terms in the minimum problems. This last procedure is in most cases superfluous. As we have pointed out, the behaviour in the normal direction is independent of the behaviour in

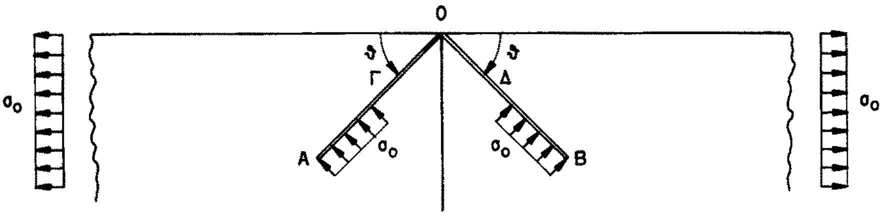


Fig. 2. The geometrical characteristics of the numerical application ($OA = OB = \alpha$, $O\Gamma = O\Delta = a/2$).

the tangential direction. However, we have the possibility to study coupled behaviour, i.e. the case in which instead of (12) and (13) we have the laws (11).

In this case we shall consider the two following subproblems:

$$\text{(i)} \quad -S_N^{\rho} \in \partial J_N(-[u_N^{(\rho)}]; S_T^{(\rho-1)}),$$
$$\text{(ii)} \quad -S_T^{(\rho+1)} \in \partial J_T(-[u_T^{(\rho+1)}]; S_N^{(\rho)}), \tag{62}$$

where in (i) [resp. in (ii)] $S_T^{(\rho+1)}$ (resp. $S_N^{(\rho)}$) is given from the previous step. The resulting algorithm is actually a fixed point algorithm; its convergence is still an open problem in the presence of crack singularities and if the general superpotential laws (62) hold [cf. also

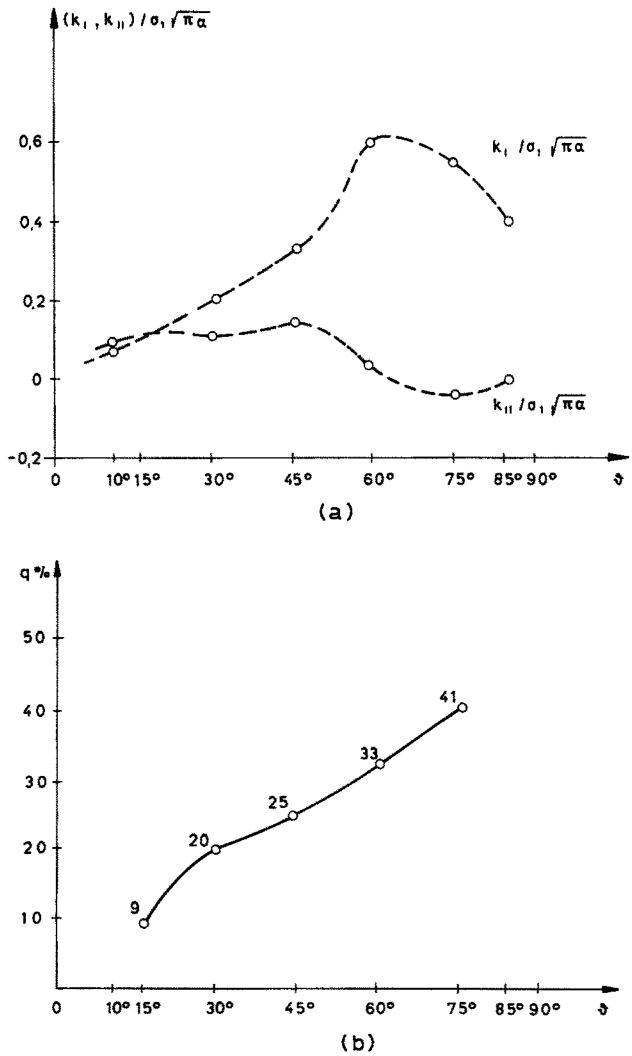


Fig. 3. Intensity factors K_I and K_{II} and the net opening of the crack.

Panagiotopoulos (1975), Nečas *et al.* (1980)]. In the case of unilateral contact with friction along the interfaces we decompose the initial problem into the following two subproblems:

(a) the pure unilateral contact problem with given tangential forces, i.e.

$$[u_N^{(\rho)}] \leqslant 0, \quad S_N^{(\rho)} \leqslant 0, \quad [u_N^{(\rho)}]S_N^{(\rho)} = 0, \quad S_{T_i}^{(\rho)} = C_{T_i}^{(\rho-1)} \quad i = 1, 2, 3; \text{ and} \tag{63}$$

(b) the pure friction problem with prescribed normal forces, i.e.

$$\begin{aligned} &\text{if } |S_T^{(\rho+1)}| < \mu |C_N^{(\rho)}| \quad \text{then} \quad [u_T^{(\rho+1)}] = 0, \\ &\text{if } |S_T^{(\rho+1)}| = \mu |C_N^{(\rho)}| \quad \text{then} \quad [u_{T_i}^{(\rho+1)}] = -\lambda S_{T_i}^{(\rho+1)}, \quad \lambda \geqslant 0, \end{aligned} \tag{64}$$

$$S_N^{(\rho+1)} = C_N^{(\rho)}. \tag{65}$$

The two subproblems (63), (64) and (65) are consecutively solved: At the p-step we solve (63) with C_{T_i} taken from the solution of (64), (65) at the $\rho-1$ step. Then the pure friction problem is solved, i.e. (64), (65) with $C_N^{(\rho)}$ taken from the solution of (63) at the ρ-step. The proof that this procedure converges to the solution of the initial problem (3)–(6) has been given in Lazaridis and Panagiotopoulos (1987), in the absence of singularities. In our case of cracks the proof is still an open problem. We can easily show that for the pure contact problem (63), (56) [resp. (57)] take the form (Lazaridis and Panagiotopoulos, 1987):

$$\min \{ \tfrac{1}{2}\beta(\lambda, \lambda) - b(\lambda) | \lambda_N \leqslant 0, \quad \lambda_T = C_T^{(\rho-1)} \quad \text{on} \quad \gamma, \quad \lambda \in L \}, \tag{66}$$

(resp.)

$$\min \{ \tfrac{1}{2}\delta([v], [v]) - d([v]) | [v_N] \leqslant 0, \quad \text{on} \quad \gamma, \quad [v] \in \Lambda \}, \tag{67}$$

whereas for the pure friction problem (64), (65), the form

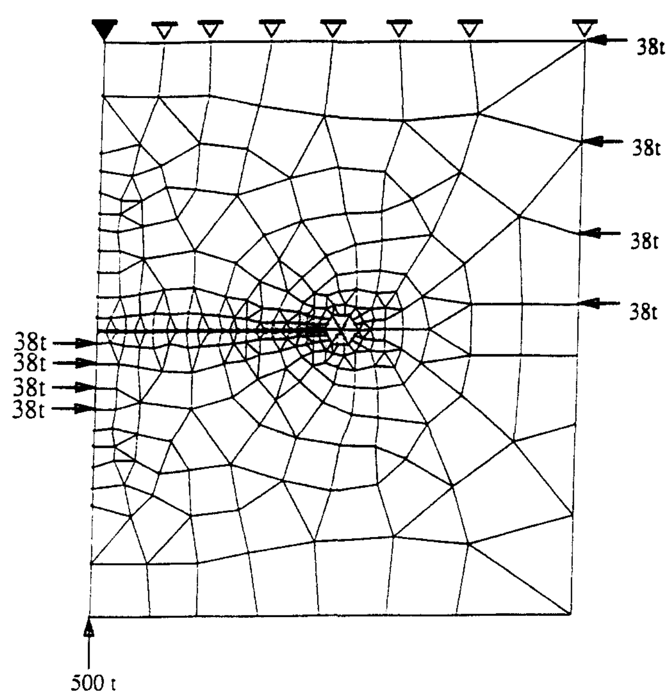
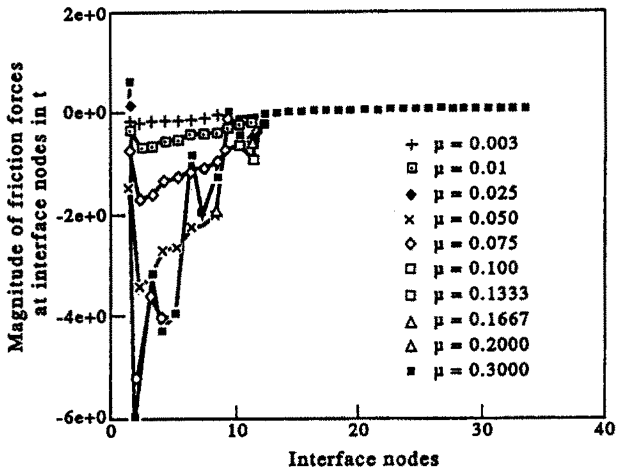
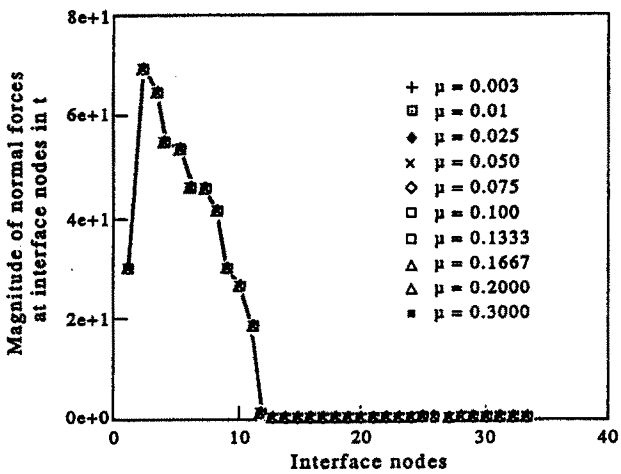


Fig. 4. Structure with a crack: length 660 mm, height 801.5 mm, crack clearance 1 mm.



(a)



(b)

Fig. 5. (a) Friction; (b) normal forces developed on the nodes of the interface for various values of the friction coefficient. In (b) all diagrams coincide.

$$\min\{\tfrac{1}{2}\beta(\lambda,\lambda)-b(\lambda)\mid|\lambda_T|\leqslant\mu|C_N^{(\rho)}|,\quad \lambda_N=C_N^{(\rho)}\quad\text{on}\quad\gamma,\quad\lambda\in L\} \tag{68}$$

(resp.)

$$\min\left\{\frac{1}{2}\,\delta([v],[v])-d([v])+\int_\gamma\mu|C_N^{(\rho)}|\,|v_T|\,\mathrm{d}\Gamma\mid[v]\in\Lambda\right\}. \tag{69}$$

Accordingly, the calculation of the stress and srain fields for problems with cracks subjected to unilateral frictional interface contact conditions is reduced to the consecutive solution of two of the above minimization problems, i.e. of (66) or (67) and of (68) or (69). Note that in the case of lack of symmetry all the above problems (66)–(69) can be formulated as linear complementarity problems (L.C.P.) with non-symmetric matrix [cf. e.g. Panagiotopoulos (1985)].

6. APPLICATIONS

As a first application (Fig. 2) we generalize the results of Tsamasphyros and Theocaris (1983) for cracks having unilateral contact with friction. In order to formulate the quadratic differentiable problems (66) or (67) and (68) we apply the unit force or relative displacement method (cf. Lazaridis and Panagiotopoulos, 1987) with respect to the boundary element scheme according to boundary element scheme of Blandford *et al.* (1981). This scheme includes quadratic boundary elements and for the treatment of the crack singularity boundary elements.

From the same paper we borrow the method for treating the singularities, as well as the procedure applied for the computation of the stress intensity factors. The matrices of the obtained minimum problems obviously have all the characteristics of the matrices resulting in classical B.E.M., i.e. they are fully populated. Due to numerical approximation they are generally non-symmetric. Note that in the present example the calculation of $\beta(.,.)$ and b can be performed analytically by applying the more accurate method described in Tsamasphyros and Theocaris (1983), which uses the Muskelishvili integrals.

In our example we have solved for the unilateral contact problem (66) and for the friction problem (68) after symmetrization. Thus we avoid the numerical solution of (69) which includes the non-differentiable absolute value term. The resulting quadratic programming problems have only a small number of unknowns and have been solved using the Hildreth and d'Esopo algorithm. Note that if the matrix is non-symmetric then a L.C.P. arises whose solution must be calculated with an appropriate algorithm for L.C.P.s [see e.g. Murty (1988)]. In Fig. 3(a) the variation of the stress intensity factors K_I, K_{II} is given for a coefficient of friction $\mu = 0.5$.

In Fig. 3(b) the length of the net opening of $O\gamma$ or $O\Delta$ as a percentage q of the crack length α is depicted. The opening always begins at O and ends towards Γ or Δ. The use of the classical B.E.M. for the calculation of the matrices of the minimization method presents some advantages in the present example due to the infinite geometry of the problem considered.

In the case of a finite structure the classical F.E.M. enriched with crack elements for the treatment of the singularities, can also be applied, for the determination of the discretized forms of the multivalued B.I.E.s. This is the case of the second example.

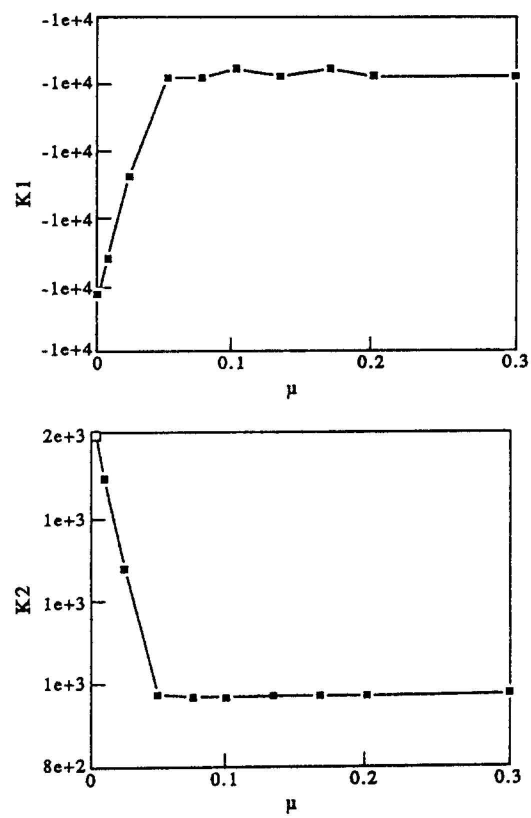


Fig. 6. The stress intensity factors K_I and K_{II} as functions of the friction coefficient μ.

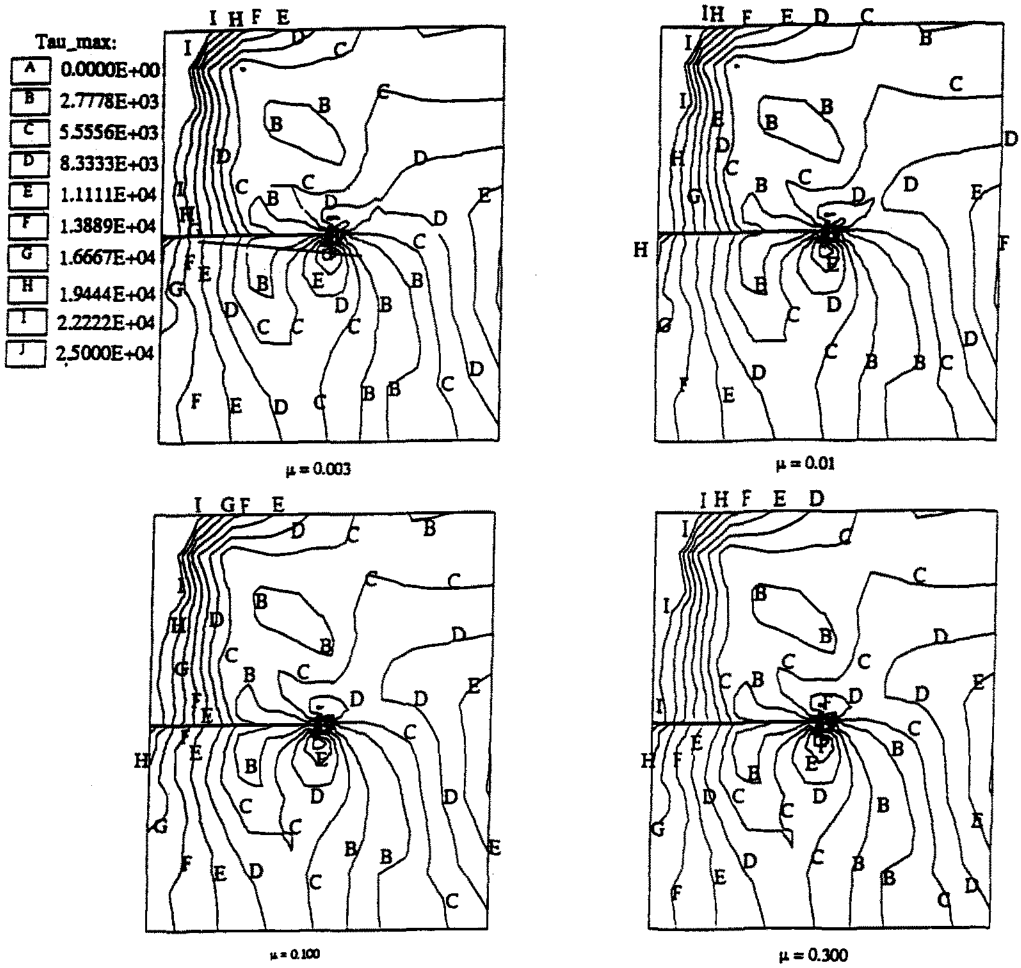


Fig. 7. Stress τ_{max} patterns for four values of the friction coefficient.

In this example we have treated an orthogonal metallic plate sized 0.67×0.8015 m with a thickness of 0.1 m, a Poisson ratio equal to 0.3 and an elasticity modulus equal to 2.1×10^7 t m^{-1}. The plate contains a central crack as in Fig. 4. The crack is assumed to have a 1 mm clearance to simulate the situation when this crack is purposefully machined. The plate was discretized in 325 plane stress elements and 33 special interface elements that simulate the unilateral contact and friction situation at the interface. The number of DOFs taking part in the unilateral contact and friction phenomenon is 124. In front of the crack tip, six singularity elements (12 node collapsed cubic isoparametric elements) take into account the singularity. The method developed by Bisbos (1990) was used and the program was run on a Hewlett-Packard 9000/750 computer. We do not intend to present exact data about computation times, but we can say that the solution of the unilateral part of the problem took about 5.2 s on the 750 and about 8.7 s on a smaller model of the same series, namely the 720. These numbers correspond to the CPU-execution time of the algorithm after the stiffness matrix has been assembled. These times were obtained without having used profile optimization of the stiffness matrix. To disperse any impressions that seem to be overwhelming in the literature about unilateral and frictional behaviour of cracks being numerically tedious, we can cite here the fact that the machine time dedicated to the editing of the graphical output for the figures at hand, was an order of magnitude more than the time needed for computations. In fact, the example represented here is considered rather small for the capabilities of the unilateral contact and friction algorithm as those are only hardware limited. To give some rough idea of the respective requirements, the algorithm requires for n contact node pairs $4n^2+60n$ words (double precision). This fact leads for 4000 nodes (i.e. 2000 contact pairs) to a memory requirement of 130 Mbytes. This example is actually a numerical study of the effect the friction coefficient μ can have on the shearing mode (II) stress intensity factor. These factors were computed by an energy method because other popular methods require the crack surface to be stress free. As one can see in Fig. 5(a) the friction forces rather rapidly converge to a stable pattern along the interface. In fact the only curves that exhibit a visible discrepancy are those for $\mu = 0.003$, 0.01, 0.025 and 0.05, i.e. the very low values of the friction coefficient. The normal forces are not affected by the friction coefficient and what one sees in Fig. 5(b) is an overlay of 10 coinciding curves. In Fig. 6 the variation of $K_{\rm I}$ and $K_{\rm II}$ against μ is depicted and one sees a rather fast stabilization of the stress intensity factors with increasing friction coefficient values. This evidence supports the discrimination of cracks to "lubricated" and "non-lubricated". Finally, in Fig. 7, the stress patterns for four values of the friction coefficient are presented. The stress patterns are quite alike but one can see the continuation of the τ-max contour lines across the interface in the sticking friction areas [to locate these areas one can observe the normal forces diagrams of Fig. 5(b)] and the dislocation of the respective stress pattern in the case of the very low friction coefficient, a fact more or less intuitively expected. What was perhaps beyond intuition was the rather quick stabilization of the stress intensity factor values with increasing friction coefficient values.

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